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MARKOV CELL STRUCTURES FOR EXPANDING MAPS IN DIMENSION TWO

BY

F. T. FARRELL AND L. E. JONES[1]

ABSTRACT. Let $f\colon M^2 \to M^2$ be an expanding self-immersion of a closed 2-manifold, then for some positive integer n, f^n leaves invariant a cell structure on M^2. A similar result is true when M is a branched 2-manifold.

0. Introduction. Let K be a topological space equipped with a continuous self-map f. It is known in many interesting cases that K can be partitioned into cells; i.e., given a cell complex structure. For example, this is possible when K is a smooth manifold. It is rarer for K to support a cell structure with respect to which f is a cellular map (i.e., leaves each skeleton invariant); for instance, this is impossible when K is the circle and f is rotation through an irrational angle.

Consider the situation when K is a closed smooth 2-manifold and f is an expanding immersion of K; i.e., $|df(X)| > |X|$ for all nonzero vectors X tangent to K and some Riemann metric on K. In this case, we show (Theorem 2.1) there is a positive integer n such that f^n (the composite of f with itself n-times) is cellular relative to some cell structure on K. (We do not know whether n can always be 1. See **[3]**, **[7]** and **[10]** for other recent interesting constructions of invariant sets.)

We prove a similar (slightly weaker) result (Theorem 3.1) for expanding immersions of compact branched 2-manifolds satisfying Axioms 1, 2 and 3^+ of Williams. (See **[14]** for the basic definitions.) These objects arise in Williams' study **[14]** of expanding attractors. We hope Theorem 3.1 will be useful in helping to understand 2-dimensional expanding attractors which are apparently more complicated (cf. **[5]**) than the 1-dimensional case where Williams has a very good theory **[13]**.

Theorems 2.1 and 3.1 are clearly extensions (in a very special setting) of the theory of Markov partitions **[1]**, **[12]**, **[2]** and **[9]**. This paper is also an introduction to the results announced in **[4]**.

1. A cellular embedding result. In this section, we formulate and prove a crucial result Lemma 1.1, a strong form of the cellular approximation

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theorem for dimension 2. It is used in §2 to construct Markov cell structures for expanding maps on 2-manifolds.

Let M^2 denote a 2-dimensional Riemannian manifold. A *special cell structure* C in M^2 is a filtration by closed subsets,

$$\varnothing = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| \subset M^2,$$

such that $C^i - C^{i-1}$ ($i = 0, 1, 2$) has finitely many connected components, called the *i-cells* of C with the following properties.

(i) The closure of each i-cell σ is homeomorphic to $\mathbf{D}^i = \{x \in \mathbf{R}^i \mid |x| \leq 1\}$ via a homeomorphism mapping σ onto $\{x \in \mathbf{R}^i \mid |x| < 1\}$. (The closure of a cell is called a closed cell.)
(ii) The intersection of two closed cells is either empty or homeomorphic to $\mathbf{D}^i$ (for some i).
(iii) Each vertex (0-cell) is contained in at least 2 and no more than 3 edges (closed 1-cells).
(iv) Edges are smooth curves.
If C satisfies only properties (i), (ii) and (iii) of (1.1), it is a *topological special cell structure*. (1.1)

For a closed subset A of M^2, we define two types of thickenings $T(A, \varepsilon)$ and $\mathfrak{T}(A, C)$ where $\varepsilon > 0$ is a real number and C is a special cell structure with $A \subset |C|$;

$$T(A, \varepsilon) = \{x \in M^2 | d(x, A) \leq \varepsilon\},$$
$$\mathfrak{T}(A, C) = \bigcup \{\sigma | \sigma \text{ a closed cell in } C, \sigma \cap A \neq \varnothing\} \quad (1.2)$$

where d denotes the metric on M^2.

Choose base points $\{P_\sigma\}$ (where $P_\sigma \in \sigma$) for the 2-cells $\{\sigma\}$ of C and let $d_0 > 0$ be the smaller of

$$d(|C^1|, \{P_\sigma\}) \quad \text{and} \quad \inf\{d(\sigma, \tau) | \sigma, \tau \text{ closed cells of } C, \sigma \cap \tau = \varnothing\}. \quad (1.3)$$

Associate sets $\{D_\sigma\}$, called auxiliary discs, to $\{\sigma\}$ satisfying

(i) D_σ is homeomorphic to $\mathbf{D}^2$,
(ii) $\sigma \subset$ interior D_σ and
(iii) $D_\sigma \subset T(\sigma, 10^{-1}d_0)$. (1.4)

Assume M^2 is compact and let $d_1 > 0$ be a number such that, for each $x \in M^2$, the exponential map is a diffeomorphism from the disc of radius d_1 centered at the origin of T_xM^2 (the tangent space to M^2 at x) to $T(x, d_1)$. As is customary, let mesh C be the maximum distance between points belonging to a common closed cell in C; recall a map $f\colon |C| \to |K|$ (between cell structures) is cellular if $f(C^i) \subset K^i$ (for each i).

LEMMA 1.1. *Let C be a special cell structure with $|C| = M^2$,* mesh $C <$ $(10)^{-1}d_1$ *and equipped with auxiliary discs $\{D_\sigma\}$; then there exists a number $\varepsilon > 0$ such that, given any other special cell structure K with $|K| = M^2$ and* mesh $K < \varepsilon$, *we can construct a cellular homeomorphism g: $|C| \to |K|$ with $g(\sigma) \subset D_\sigma$ for each closed cell σ in C and so that $g(\omega)$ contains a vertex of K for each open 1-cell ω in C.*

The proof of this result occupies most of §1. Pick a number $d_2 > 0$ such that $T(\sigma, 2d_2) \subset D_\sigma$ for each closed cell σ in C and satisfying the following extra constraint. For each edge ω in C and vertex v contained in ω, ω and the boundary of $T(v, r)$ intersect transversally in a single point provided $0 < r \leqslant 2d_2$. Consequently, we can smoothly parameterize each edge ω as a function ω: $[0, 3] \to M^2$ with the following properties

(i) $d\omega(t)/dt \neq 0$ for $t \in [0, 3]$,
(ii) $\omega([0, 1]) \subset T(\omega(0), 3d_2/2)$,
(iii) $\omega([2, 3]) \subset T(\omega(3), 3d_2/2)$ and
(iv) $\omega((1, 2)) \subset M^2 - \cup \{T(v, 3d_2/2)|v$ a vertex in $C\}$. (1.5)

(Fix such a choice of parameterizations for the remainder of §1.)

The construction of g uses the following elementary fact. (Its verification is left as an exercise.)

LEMMA 1.2. *If A is a closed connected subset of M^2 and K is a special cell structure with $|K| = M^2$, then $\mathfrak{T}(A, K)$ is connected; in fact, any two vertices v_0 and v_1 in $\mathfrak{T}(A, K)$ can be joined by a simple polygonal arc in $\mathfrak{T}(A, K)$.*

(A *polygonal arc* is a concatenation of edges in a complex.)

The ε posited in Lemma 1.1 is any number smaller than $d_2/3$ satisfying

(i) $T(\omega[0, 1], \varepsilon) \subset T(\omega(0), 2d_2)$,
(ii) $T(\omega[2, 3], \varepsilon) \subset T(\omega(3), 2d_2)$,
(iii) $T(\omega[1, 2], \varepsilon) \subset M^2 - \cup \{T(v, d_2)|v$ a vertex in $C\}$,
(iv) $T(\omega_1, \varepsilon) \cap T(\omega_2, \varepsilon) \subset T(v, d_2)$ if $\omega_1 \cap \omega_2 = v$, (1.6)

where ω, ω_1, ω_2 are (parameterized) edges and v is a vertex of C.

First construct $g|C^1$; for each edge ω, we must determine $g(\omega)$. As an approximation to $g(\omega)$, we construct simple polygonal arcs ω': $[0, 3] \to \mathfrak{T}(\omega, K)$ with the following properties

(i) $\omega'[0, 1] \subset T(\omega(0), 2d_2)$,
(ii) $\omega'[2, 3] \subset T(\omega(3), 2d_2)$,
(iii) $\omega'[1, 2] \subset M^2 - \cup \{T(v, d_2)|v$ a vertex of $C\}$,
(iv) $\omega'[0, 3] \subset T(\omega, 2d_2)$ and
(v) if $\omega_1(0) = \omega_2(0)$ $(\omega_1(0) = \omega_2(3))$, then $\omega_1'(0) = \omega_2'(0)$ $(\omega_1'(0) = \omega_2'(3))$, (1.7)

where ω_1, ω_2 are edges in C.

To construct ω', pick 4 vertices v_i $(i = 0, 1, 2, 3)$ from K with $v_i \in \mathfrak{T}(\omega(i), K)$; if α is a second edge in C with $\alpha(0) = \omega(0)$, make the same choice of v_0 in constructing α'. (If $\alpha(0) = \omega(3)$, then v_0 for α' should be the v_3 chosen for ω'). Now, using Lemma 1.2 connect successive vertices v_i, v_{i+1} by simple polygonal arcs γ_i in $\mathfrak{T}(\omega[i, i+1], K)$; the concatenation of these form a polygonal arc in $\mathfrak{T}(\omega[0, 3], K)$ connecting v_0 to v_3. The result may not be a simple arc; but, it is easy to find subarcs γ_i' of γ_i which concatenate to form a simple arc ω' connecting v_0 to v_3. (Note v_1 and v_2 need not be points on ω'.) The collection $\{\omega'\}$ thus constructed can be parameterized to satisfy (1.7).

Note, if ω_1 and ω_2 are nonintersecting edges of C, then $\omega_1'[0, 3]$ does not intersect $\omega_2'[0, 3]$. Unfortunately, when ω_1 and ω_2 are distinct but share a common vertex, possibly ω_1' and ω_2' meet in more than a common endpoint. However, by an elementary combinatorial argument, this particular collection of $\{\omega'\}$ (constructed above) can be modified to form a new collection $\{\omega''\}$ of simple polygonal arcs having the following properties

(i) $\{\omega''\}$ satisfies (1.7),
(ii) $\omega''(t) = \omega'(t)$ for $t \in [1, 2]$ and
(iii) $\omega''[0, 3] \cap \alpha''[0, 3]$ contains at most one point, if ω and α are distinct edges in C. (1.8)

We leave this argument as an exercise. Hint. The $\{\omega'\}$ can be chosen so that $\bigcup\{\omega''[0, 3]\} \subset \bigcup\{\omega'[0, 3]\}$ but neither of the following pair of equations need be valid

$$\omega''(0) = \omega'(0), \qquad \omega''(3) = \omega'(3).$$

Figure 1 shows the hinted modification. In it, piecewise smooth curves are used instead of polygonal arcs for purposes of illustration. The dashed lines in the second picture indicate parts of $\bigcup\{\omega_i'\}$ deleted in forming $\{\omega_i''\}$; the large circle in both pictures is the boundary of $T(x, d_2)$ where $x = \omega_1(0)$ or $\omega_1(3)$ as the case requires.

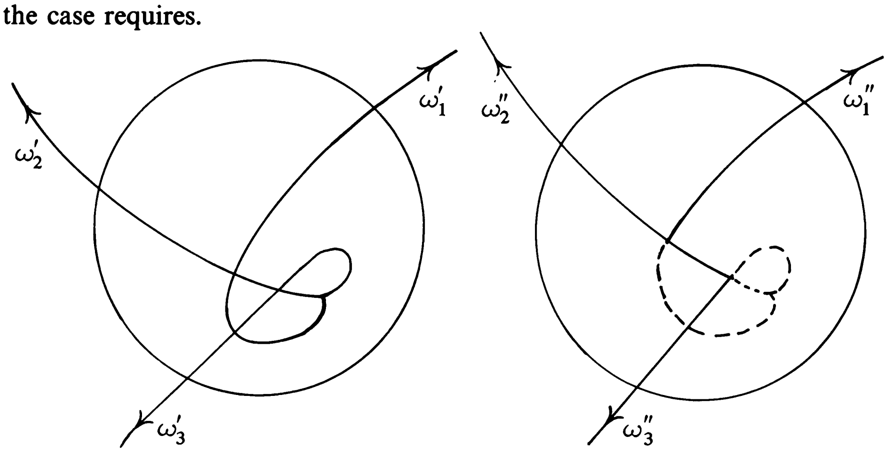

FIGURE 1

Now define $g|C^1$ by the formula $g(\omega(t)) = \omega''(t)$ for $t \in [0, 3]$ where ω is a (parameterized) edge; because of (1.8), this map is an embedding and $g(\omega) \subset D_\omega$. (Since mesh $K < d_2/3$, $g(\omega)$ contains a vertex of K for each open 1-cell ω in C.) In fact, if σ is a closed 2-cell and $\partial\sigma$ denotes its boundary, then $g(\partial\sigma) \subset D_\sigma$; hence, by Schoenflies' Theorem (cf. [8, p. 175]), $g|\partial\sigma$ can be extended to a homeomorphism of σ onto the closure of the interior component of $D_\sigma - g(\partial\sigma)$. In this way, extend $g|C^1$ to a cellular map g from C to K with $g(\sigma) \subset D_\sigma$ for each closed cell σ in C. This last fact (cf. (1.4)) implies

$$g|C^1\colon C^1 \to M^2 - \{P_\sigma\} \tag{1.9}$$

is homotopic to the inclusion map; hence, by an elementary winding number argument, $g\colon M^2 \to M^2$ is a homeomorphism. This completes the proof of Lemma 1.1.

In §3 a relative version of this result is used. We now formulate it leaving its proof as an exercise.

Drop the compactness constraint on M^2 and consider the following condition for special cell structures C in M^2.

For each $x \in |C|$, the exponential map is a diffeomorphism from the disc of radius 10 mesh C centered at the origin of T_xM^2 to $T(x, 10 \text{ mesh } C)$. (1.10)

LEMMA 1.3. *Let C be a special cell structure in M^2 satisfying* (1.10) *and equipped with auxiliary discs $\{D_\sigma\}$ (cf.* (1.4)); *let $U \subset V \subset C$ be subcomplexes such that any closed cell of C which meets $|U|$ is contained in $|V|$. Then there exists a second collection of auxiliary discs $\{D'_\sigma\}$ (cf.* (1.4)) *and a real number $\varepsilon > 0$ such that, for any other special cell structure K in M^2 with* mesh $K < \varepsilon$ *and $T(|C|$, mesh $C) \subset |K|$ and any cellular embedding $h\colon |V| \to |K|$ satisfying $h(\sigma) \subset D'_\sigma$ for all cells σ in V, we can construct a cellular embedding g: $|C| \to |K|$ such that*

(i) *$g(x) = h(x)$ for $x \in |U|$,*
(ii) *$g(\sigma) \subset D_\sigma$ for all cells σ in C and*
(iii) *$g(\omega)$ contains a vertex of K for each open* 1-*cell ω in C.*

2. Markov cell structures (2-manifold case). Let M^2 be a closed 2-manifold equipped with a map $f\colon M^2 \to M^2$; a *Markov cell structure* for (M^2, f) is a topological special cell structure $\mathcal{C}$ in M (cf. (1.1)) with $|\mathcal{C}| = M$ and such that $f^n\colon |\mathcal{C}| \to |\mathcal{C}|$ is cellular for some positive integer n. This section is devoted to proving the following result.

THEOREM 2.1. *If $f\colon M^2 \to M^2$ is an expanding endomorphism on a closed 2-manifold, then (M^2, f) has a Markov cell structure.*

We note Shub [11] has shown that, under the hypotheses of Theorem 2.1,

M^2 is either the torus on the Klein bottle and f is topologically conjugate to a linear expanding map; but we will not use these facts in proving Theorem 2.1.

To prove this result, start by choosing a special cell structure C on M^2 with $|C| = M^2$ and mesh $C < d_1/5$. (See the paragraph preceding the statement of Lemma 1.1 for the definition of d_1.) Such a complex C can be constructed (for example) by using a dual cell structure to a smooth triangulation of M^2 having sufficiently small mesh.

Choose a collection of base points $\{P_\sigma\}$ and auxiliary discs $\{D_\sigma\}$ for C (cf. (1.4)) and let $\varepsilon > 0$ be the number posited in Lemma 1.1. Let n be a positive integer sufficiently large that

$$|df^n(X)| > \varepsilon^{-1}(\text{mesh } C)|X| \text{ and } 2|X| \tag{2.1}$$

for each nonzero vector X tangent to M^2 and let F denote f^n. Since F: $M^2 \to M^2$ is a covering space, we can form a new special cell structure K with $|K| = M^2$ by setting $K^i = F^{-1}(C^i)$; note mesh $K < \varepsilon$ because of (2.1). Now, applying Lemma 1.1 to this set up, we obtain a cellular homeomorphism g: $|C| \to |K|$ with $g(\sigma) \subset D_\sigma$ for each closed cell σ of C. Since $d(x, g(x)) < d_1$ for all $x \in M^2$ and F is expanding, we can (by lifting g via F^s) form a sequence of homeomorphisms g_s: $M^2 \to M^2$ (indexed by the nonnegative integers) having the following properties

$$\begin{aligned} &\text{(i) } Fg_s = g_{s-1}F \text{ for } s > 0,\\ &\text{(ii) } g_0 = g \text{ and}\\ &\text{(iii) } d(x, g_s(x)) \leqslant 2^{-s}d_1 \text{ for all } x \in M^2; \end{aligned} \tag{2.2}$$

in fact, x and $g_s(x)$ are in the same component of $F^{-s}T(F^s(x), d_1)$. Form the composites $G_s = g_s g_{s-1} \cdots g_0$; because of (2.2), the sequence G_s converges uniformly to a continuous function G: $M^2 \to M^2$ satisfying the equation

$$FG = GFg. \tag{2.3}$$

Define $\mathcal{C}^i$, closed subsets of M^2, by $\mathcal{C}^i = G(C^i)$; because of (2.3) and the fact Fg: $|C| \to |C|$ is cellular, we have

$$F(\mathcal{C}^i) \subset \mathcal{C}^i \quad \text{for all } i. \tag{2.4}$$

To complete the demonstration of Theorem 2.1, it remains to show that the filtration $\mathcal{C}$ (defined above) is a topological special cell structure. (Since G is homotopic to the identity map, we note $|\mathcal{C}| = G(M^2) = M^2$.) We will accomplish this by constructing a homeomorphism G': $M^2 \to M^2$ with $G'(C^i) = G(C^i)$ for all i; the construction of G' occupies the remainder of this section.

Define a sequence of special cell structures $C(s)$ by the equations $C(s)^i = F^{-s}(C^i)$; note $C(0) = C$ and $C(1) = K$. Next, construct a sequence of cellular homeomorphisms h_s: $C(s)^1 \to C(s)^1$ with $h_s(\sigma) = \sigma$ for each vertex or edge σ of $C(s)$ and having the following property.

If ω is an edge of $C(s)$ parameterized by arc length t, then $g_s h_s \omega(\Omega/2)$ is a vertex of $C(s+1)$ (Ω denotes the total arc length of ω) and the derivative of the arc length of $g_s h_s \omega(t)$ with respect to t exists except when $t = \Omega/2$ and is constant on $(0, \Omega/2)$ and $(\Omega/2, \Omega)$. (2.5)

It is easy to construct such homeomorphisms using the fact that g_s: $|C(s)| \to |C(s+1)|$ is cellular and $g_s(\sigma)$ contains a vertex of $C(s+1)$ for each *open* 1-cell σ of $C(s)$.

Define a sequence of embeddings H_i by the equations

$$H_i = g_i h_i g_{i-1} h_{i-1} \cdots g_0 h_0; \tag{2.6}$$

it follows in a straightforward fashion that this sequence converges uniformly to a continuous function H: $C^1 \to M^2$ such that $H(C^i) = G(C^i)$ for $i = 0$ and 1.

LEMMA 2.2. *The function H: $C^1 \to \mathcal{C}^1$ is a homeomorphism.*

PROOF. It remains only to show that H is one-to-one. Let x and y be distinct points in C^1 and choose a simple piecewise smooth arc α: $[0, 1] \to C^1$ with $x = \alpha(a)$ and $y = \alpha(b)$ where $0 < a < b < 1$ (cf. Lemma 1.2). Because of (2.5) there exists an integer $s > 0$ such that the simple arc $H_s\alpha$ contains at least four vertices v_i $(i = 1, 2, 3, 4)$ of $C(s+1)$ with $v_i = H_s\alpha(t_i)$ where $t_1 < a < t_2 < t_3 < b < t_4$; in particular, $H_s(x)$ and $H_s(y)$ belong to nonintersecting edges ω_x and ω_y of $C(s+1)$. From this remark, it is easily seen (use the defining properties of h_i) that $H(x) \in \hat{G}_{s+1}(\omega_x)$ and $H(y) \in \hat{G}_{s+1}(\omega_y)$ where $\hat{G}_{s+1}$ is the composite GG_s^{-1}; hence, Lemma 2.2 is a consequence of the following result.

LEMMA 2.3. *If σ_1 and σ_2 are nonintersecting closed cells in $C(s)$, then $\hat{G}_s(\sigma_1) \cap \hat{G}_s(\sigma_2) = \varnothing$.*

PROOF. First observe, using Lemma 1.1, 1.4, 2.1, that $G(\sigma) \subset T(\sigma, 5^{-1}d_0)$ for each closed cell σ of C; also using (2.2), that $F^s\hat{G}_s = GF^s$; hence if $F^s\sigma_1 \cap F^s\sigma_2 = \varnothing$, then $\hat{G}_s(\sigma_1) \cap \hat{G}_s(\sigma_2) = \varnothing$. On the other hand, if $F^s\sigma_1$ and $F^s\sigma_2$ intersect, then $T(F^s\sigma_1 \cup F^s\sigma_2, 5^{-1}d_0) \subset T(x, d_1)$ for some point $x \in M^2$. Since $T(x, d_1)$ is homeomorphic to a closed 2-disc, it is evenly covered by F^s; in particular, $\sigma_i \subset S_i$ $(i = 1, 2)$ where S_1 and S_2 are distinct components of $F^{-s}T(x, d_1)$. Finally observe $\sigma_i \cup \hat{G}_s\sigma_i \subset S_i$ $(i = 1, 2)$; hence $\hat{G}_s\sigma_1$ and $\hat{G}_s\sigma_2$ cannot intersect.

Because of Lemma 2.2 (and the paragraph preceding it) we define $G'|C^1$ to be H. Let σ be a closed 2-cell of C with boundary denoted by $\partial\sigma$; then $H(\partial\sigma)$ is contained in the interior of $T(P_\sigma, d_1)$ since

$$H(\partial\sigma) = G(\partial\sigma) \subset G(\sigma) \subset T(\sigma, 5^{-1}d_0). \tag{2.7}$$

Since $\partial\sigma$ is homeomorphic to a circle, $T(P_\sigma, d_1)$ to a closed 2-disc and H is an embedding, Schoenflies' Theorem allows us to extend $G'|\partial\sigma$ to a homeomorphism of σ to the closure of the interior component of $T(P_\sigma, d_1) - G'(\partial\sigma)$; thus, we extend $G'|C^1$ to all of M^2.

To complete the demonstration of Theorem 2.1, it remains to verify that G' is a one-to-one function. (Note any one-to-one continuous self-map of a closed connected manifold must be onto.) To do this, first observe that $G'(\sigma) \subset G(\sigma)$ for each closed 2-cell σ of C; this is a consequence of the fact that $G'|\partial\sigma$ and $G|\partial\sigma$: $\partial\sigma \to G'(\partial\sigma)$ are homotopic (hence $G|\partial\sigma$ is essential). Therefore, it suffices to show for all pairs of distinct 2-cells σ_1 and σ_2 in C that the following containment is true

$$G(\sigma_1) \cap G(\sigma_2) \subset G'(\sigma_1 \cap \sigma_2). \tag{2.8}$$

Let $z \in G(\sigma_1) \cap G(\sigma_2)$; hence, there exist points $x \in \sigma_1$ and $y \in \sigma_2$ with $G(x) = z = G(y)$; consequently, $d(G_i(x), G_i(y)) \to 0$ as $i \to \infty$. Since G_i is a homeomorphism, there must exist points $x_i \in \partial\sigma_1$ and $y_i \in \partial\sigma_2$ such that $G_i(x_i) \to z$ and $G_i(y_i) \to z$ as $i \to \infty$ (cf. Figure 2); but $\partial\sigma_i$ $(i = 1, 2)$ is compact; hence $z \in G(\partial\sigma_1) \cap G(\partial\sigma_2)$.

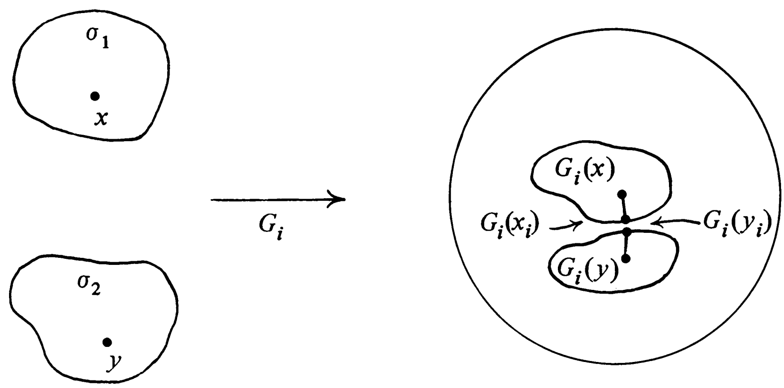

FIGURE 2

Since $G(\partial\sigma_i) = H(\partial\sigma_i)$ $(i = 1, 2)$ and H is a one-to-one function (Lemma 2.2), we have

$$z \in H(\partial\sigma_1 \cap \partial\sigma_2) = G'(\sigma_1 \cap \sigma_2) \tag{2.9}$$

which verifies (2.8) and completes the proof of Theorem 2.1.

REMARK 2.4. Let f: $T^2 \to T^2$, where T^2 is the torus, be an expanding endomorphism induced by a 2×2 matrix A with integral entries and whose eigenvalues are real but irrational numbers having distinct absolute values. It is implicit in Franks' paper **[6]** that a proper closed f-invariant subset of T^2 cannot contain a C^1-arc; i.e., a continuously differentiable arc. In particular, if

$$A = \begin{pmatrix} 0 & 11 \\ -1 & 7 \end{pmatrix},$$

then the 1-skeleton of the Markov cell structure constructed in Theorem 2.1 for (T^2, f) cannot contain a C^1-arc. (See also Bowen's paper [3].)

3. Markov cell structures (branched 2-manifold case). This section is devoted to proving an analogue of Theorem 2.1 in the branched 2-manifold case. (See Williams' paper [14] for the basic definitions and facts concerning branched manifolds.) A *cell structure* C for a compact branched 2-manifold M^2 is a filtration of M^2 by closed subsets

$$\emptyset = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| = M^2 \tag{3.1}$$

such that $C^i - C^{i-1}$ $(i = 0, 1, 2)$ has a finite number of components, called i-cells, satisfying property (i) in (1.1). If M^2 is equipped with a map f, a *Markov cell structure* for (M^2, f) is a cell structure C for M^2 such that f^n: $|C| \to |C|$ is cellular for some positive integer n.

THEOREM 3.1. *Let f: $M^2 \to M^2$ be an immersion satisfying Axioms* 1, 2 *and* 3^+ *of* [14] *where M^2 is a compact branched 2-manifold, then (M^2, f) is shift equivalent to a pair (N^2, g) having a Markov cell structure.*

The proof of Theorem 3.1 is similar to, but technically more complicated than, that of Theorem 2.1; hence, we sketch it, going into detail only where it differs from the previous argument.

By [14, Lemma 5.6], (M^2, f) is shift equivalent to a pair (N^2, g) where N^2 is normally branched; hence, we will assume that M^2 is normally branched. Let βM denote the branch set of M^2; as a consequence of [14, §8], M^2 contains a 2-dimensional compact submanifold (with boundary) A and a 2-dimensional compact branched submanifold (with boundary) B' satisfying

$$\begin{aligned} &\text{(i) interior } A \cup \text{interior } B' = M^2 \text{ and} \\ &\text{(ii) } \beta M \subset \text{interior } B' \cap (M^2 - A); \end{aligned} \tag{3.2}$$

furthermore, there is a compact 2-manifold B (with boundary), a surjective immersion p: $B' \to B$ which maps βM homeomorphically onto $p(\beta M)$, and an immersion φ: $B \to M^2$ with $f(x) = \varphi p(x)$ for all $x \in B'$. By deleting short collar neighborhoods from A, B and B', we obtain compact submanifolds A_i, B_i and B_i' $(i = -1, 0, 1)$ with

$$\begin{aligned} &A_{-1} \subset \text{interior } A_0 \subset \text{interior } A_1 \subset \text{interior } A \quad \text{and} \\ &B_{-1} \subset \text{interior } B_0 \subset \text{interior } B_1 \subset \text{interior } B \end{aligned} \tag{3.2.1}$$

such that $B_i' = p^{-1}B_i$ and (3.2) is satisfied when A and B' are replaced by A_{-1} and B'_{-1}. We also assume that the Riemannian metrics on M^2, A, B' and B fit together consistently; i.e., A and B' have the metrics induced from M^2

and dp: $TB' \to TB$ preserves the Riemann metric.

A (*topological*) *branched cell structure* $\mathfrak{B}$ for M^2 is a filtration by closed subsets

$$\varnothing = \mathfrak{B}^{-1} \subset \mathfrak{B}^0 \subset \mathfrak{B}^1 \subset \mathfrak{B}^2 = |\mathfrak{B}| = M^2 \tag{3.3}$$

such that there exist (topological) special cell structures C in the interior of A and K in the interior of B satisfying the following conditions

(i) $\mathfrak{B}^i = C^i \cup p^{-1}K^i$,
(ii) $A_{-1} \subset |C|$, $B_{-1} \subset |K|$ and
(iii) $|C| \cap p^{-1}|K|$ is a subcomplex of C; each cell of which maps homeomorphically via p onto a cell of K. (3.4)

The components of $\mathfrak{B}^i - \mathfrak{B}^{i-1}$ are (in general) not cells when they intersect βM^2. A map f: $|\mathfrak{B}| \to |\mathfrak{B}|$ is *cellular* if $f(\mathfrak{B}^i) \subset \mathfrak{B}^i$ (for all i). We will derive Theorem 3.1 directly from the following result.

LEMMA 3.2. *There is a topological branched cell structure* $\mathfrak{B}$ *for* M^2 *of arbitrarily small mesh such that* f^n: $|\mathfrak{B}| \to |\mathfrak{B}|$ *is a cellular map for some positive integer* n.

Before proving this lemma, we use it together with the collapsing technique introduced by Williams (cf. [**14**, Lemmas 2.2 and 5.4]) to complete the demonstration of Theorem 3.1. Let K be the topological special cell structure for B posited in (3.4). Since $\mathfrak{B}$ can be constructed with arbitrarily small mesh, we can assume that K has a subcomplex K_0 such that

$$p(\beta M^2) \subset \text{interior } |K_0| \subset \text{interior } B_{-1}. \tag{3.5}$$

Form N^2 by collapsing $p^{-1}|K_0|$ under the immersion p: $B' \to B$ and let g: $N^2 \to N^2$ be the immersion induced from f, then (N^2, g) is shift equivalent to (M^2, f) and $\mathfrak{B}$ induces a cell structure on N^2 (via the canonical quotient map) with respect to which g^n is cellular; i.e., (N^2, g) has a Markov cell structure.

REMARK 3.3. The Markov cell structure constructed above may fail to satisfy property (ii) in (1.1); for instance, if K_0 is not a "full" subcomplex in K, this is possible. But, by constructing more carefully, we believe this property can also be satisfied.

It remains to prove Lemma 3.2; we start by constructing a branched cell structure $\mathfrak{B}(0)$ for M^2 of arbitrarily small mesh. First, triangulate M^2 by a smooth triangulation $\mathfrak{T}_1$ (with arbitrarily small mesh) so that B' is a subcomplex with its triangulation induced via p^{-1} from a smooth triangulation $\mathfrak{T}_2$ of B (cf. [**14**, Lemma 5.7]). Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be subcomplexes consisting of all closed simplices from $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively, which meet A_0 and B_1, respectively; if mesh $\mathfrak{T}_1$ is sufficiently small, then

$$|\mathfrak{F}_1| \subset \text{interior } A_1 \quad \text{and} \quad |\mathfrak{F}_2| \subset \text{interior } B. \tag{3.6}$$

Let C and K be the dual cell complexes to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively; then C, K satisfy conditions (ii) and (iii) of (3.4) and hence determine via formula (i) of (3.4) a branched cell structure $\mathfrak{B}(0)$ for M^2 with arbitrarily small mesh. (Note also $B_1 \subset |K|$ and $|C| \subset A_1$.)

We assume that

$$|df(X)| > 2|X| \tag{3.7}$$

for each vector X tangent to M^2. (To do this, we may have to replace f by a power of itself.) Define a second branched cell structure $\mathfrak{B}(1)$ for M^2 by letting $\mathfrak{B}(1)^i = f^{-1}(\mathfrak{B}(0)^i)$. (For this to be a branched cell structure, mesh $\mathfrak{B}(0)$ must be sufficiently small.) There are special cell structures $C(1)$ in the interior of A and $K(1)$ in the interior of B satisfying (3.4) (with $\mathfrak{B}$, C and K replaced by $\mathfrak{B}(1)$, $C(1)$ and $K(1)$) and having the following properties

(i) $C(1)^i \subset \mathfrak{B}(1)^i$ and $K(1)^i \subset \varphi^{-1}(\mathfrak{B}(0)^i)$;
(ii) $T(|C|, \text{mesh } \mathfrak{B}(0)) \subset |C(1)|$ and $T(|K|, \text{mesh } \mathfrak{B}(0)) \subset |K(1)|$. (3.8)

(In order to satisfy property (ii), mesh $\mathfrak{B}(0)$ must again be sufficiently small.)

Let $\{P_\sigma\}$, $\{D_\sigma\}$ be base points and auxiliary discs for C; let $V = p^{-1}|K| \cap |C|$ and $U = p^{-1}\mathfrak{T}(B_0, K) \cap |C|$. If mesh $\mathfrak{B}(0)$ is sufficiently small, then $U \subset V \subset C$ satisfy the first hypothesis of Lemma 1.3; let $\{D'_\sigma\}$ be the auxiliary discs posited there. Replacing these by smaller discs (if necessary), we may assume there are auxiliary discs $\{\hat{D}_\tau\}$ for K such that $p(D'_\sigma) = \hat{D}_\tau$ for each closed cell σ of V and τ of K with $p(\sigma) = \tau$. Also let $\{\hat{P}_\tau\}$ be a collection of base points for K such that $p(P_\sigma) = \hat{P}_\tau$ for 2-cells σ of V and τ of K with $p(\sigma) = \tau$. Replacing f by a high power of itself (if necessary), we can assume that mesh $\mathfrak{B}(1)$ is small enough so that we can apply Lemma 1.3 (twice) to obtain cellular embeddings $\hat{g}\colon |K| \to |K(1)|$ and $g\colon |C| \to |C(1)|$ with the following properties

(i) $pg(x) = \hat{g}p(x)$ for $x \in |C| \cap p^{-1}|K(0)|$ where $K(0) = \mathfrak{T}(B_0, K)$;
(ii) $g(\sigma) \subset D_\sigma$ and $\hat{g}(\tau) \subset \hat{D}_\tau$ where σ and τ are closed cells in C and K, respectively. (3.9)

Of course, mesh $\mathfrak{B}(0)$ must be small enough that Lemma 1.3 applies; in particular, $T(x, 5c)$ should be homeomorphic to a closed 2-disc for both $x \in A_0$ and $x \in B_0$ where $c = 2$ mesh $\mathfrak{B}(0)$; also, assume $T(A_1, c) \subset A$ and $T(B_1, c) \subset B$. Now, define inductively a sequence of embeddings $g_s\colon A_1 \to A$ and $\hat{g}_s\colon B_1 \to B$ as follows. If $x \in A_1$ and $f(x) \in A_0$ (respectively, $f(x) \in B'_0$), let $g_s(x)$ be the unique point in the same component of $f^{-1}T(f(x), c)$, $(f^{-1}p^{-1}T(pf(x), c))$ as x such that

$$fg_s(x) = g_{s-1}f(x), \qquad (pfg_s(x) = \hat{g}_{s-1}pf(x)); \tag{3.10.1}$$

if $x \in B_1$ and $y = \varphi(x) \in A_0$ (respectively, $y \in B_0'$), let $\hat{g}_s(x)$ be the unique point in the same component of $\varphi^{-1}T(y, c)$, $(\varphi^{-1}p^{-1}T(p(y), c))$ as x such that

$$\varphi\hat{g}_s(x) = g_{s-1}(y), \qquad (p\varphi\hat{g}_s(x) = \hat{g}_{s-1}p(y)). \tag{3.10.2}$$

(When $s = 1$ use g and $\hat{g}$ in formulas 3.10.1, 3.10.2 in place of g_0 and $\hat{g}_0$, respectively.) Note the following analogue of 3.9(i) is true

$$pg_s(x) = \hat{g}_s p(x) \quad \text{for } x \in A_1 \cap B_1'. \tag{3.10.3}$$

Next, form the composite embeddings G_s: $|C(0)| \to A$ where $C(0) = \mathfrak{T}(A_0, C)$ and $\hat{G}_s$: $|K(0)| \to B$ defined by the equations

$$G_s(x) = g_s g_{s-1} \cdots g_1 g(x) \quad \text{for } x \in |C(0)| \qquad \text{and}$$
$$\hat{G}_s(y) = \hat{g}_s\hat{g}_{s-1} \cdots \hat{g}_1 \hat{g}(y) \quad \text{for } y \in |K(0)|. \tag{3.11}$$

These sequences converge uniformly to continuous functions G: $|C(0)| \to A$ and $\hat{G}$: $|K(0)| \to B$ satisfying

$$pG(x) = \hat{G}p(x) \quad \text{for } x \in |C(0)| \cap p^{-1}|K(0)|. \tag{3.12}$$

Define filtrations $\mathcal{C}$ and $\mathcal{K}$ by the equations

$$\mathcal{C}^i = G\big(C(0)^i\big) \quad \text{and} \quad \mathcal{K}^i = \hat{G}\big(K(0)^i\big); \tag{3.13}$$

although G and $\hat{G}$ need not be embeddings, arguing as in §2, it can be shown that $\mathcal{C}$ and $\mathcal{K}$ are topological special cell structures in the interiors of A and B, respectively. Also, conditions (ii) and (iii) of (3.4) are satisfied when C and K are replaced by $\mathcal{C}$ and $\mathcal{K}$; hence, letting $\mathcal{B}^i = \mathcal{C}^i \cup p^{-1}\mathcal{K}^i$, we obtain a topological branched cell structure $\mathcal{B}$ for M^2 and, using (3.10.1), (3.10.2), it can be shown that f: $|\mathcal{B}| \to |\mathcal{B}|$ is cellular. This completes the proof of Lemma 3.2.

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Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Department of Mathematics, SUNY Center at Stony Brook, Stony Brook, New York 11794

Current address (F. T. Farrell): Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Current address (L. E. Jones): Department of Mathematics, Yale University, New Haven, Connecticut 06520